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LETTER TO THE EDITOR

The spectrum of the period-doubling operator in terms of cycles

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Abstract. The eigenvalue spectrum of the period-doubling operator is evaluated both in terms of unstable periodic orbits and by a diagonalization at the operator. Cycles up to length ten yield the twelve leading eigenvalues; direct diagonalization yields some 50 eigenvalues. The Feigenbaum δ, the Hausdorff dimension and the escape rate for the period-doubling repeller are evaluated to high accuracy.

In this letter we apply the technique for the extraction of correlation exponents of [1] to evaluation of the spectrum of the period-doubling renormalization operator. The basic idea of relating the Feigenbaum constant δ to the scaling (or the presentation) function is due to Sullivan [2], and is implemented as an eigenvalue spectrum calculation in [3–5]. We refer the reader to [3, 4] for an introduction to the cycle expansions in general, and their application to the evaluation of the stability, the dimension and the escape rate of the period-doubling repeller in particular. The Fredholm determinant evaluation method and the transfer operator diagonalization [1] used here are superior to the method of locating zeros of finite products of dynamical ζ functions [6] of [3, 4, 7, 8], and in that sense the present letter supersedes the above references.

The repeller we study here is the non-wandering set of the period-doubling presentation function [9]

$$\begin{aligned} f_0(x) &= \alpha g(x) \qquad & g(\alpha^{-1}) \leqslant x \leqslant 1 \\ f_1(x) &= \alpha x \qquad & \alpha^{-1} \leqslant x \leqslant \alpha^{-2} \end{aligned} \tag{1}$$

where $g(x)$ is the universal period-doubling function which satisfies

$$g(x) = \alpha g \circ g(x/\alpha). \tag{2}$$

The stability [3] of a repeller is probed by perturbing its points by an infinitesimal smooth perturbation $x \to x + h(x)$, $|h(x)| \ll 1$, and investigating the growth of the perturbation under iterations of the mapping. In one iteration the perturbation $h(y)$ expands to $f'(y)h(y)$; the total perturbation at the point x is the sum of the perturbations at its pre-images:

$$(\mathscr{L} \circ h)(x) = \sum_{y=f^{-1}(x)} f'(y)h(y) = \int \mathrm{d}y\, \delta(y - f^{-1}(x)) f'(y) h(y). \tag{3}$$

This relation defines the 'transfer operator' $\mathscr{L}(y, x) = \delta(y - f^{-1}(x)) f'(y)$. The period-doubling fixed point linear stability equation [10] is obtained by substituting $g(x) \to g(x) + h_n(g(x))$ into (2):

$$h_{n-1}(g(x)) = \alpha g'(g(x/\alpha)) h_n(g(x/\alpha)) + \alpha h_n(g(x)/\alpha).$$

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Recast [4, 11] into the presentation function form (1), this is a stability perturbation of the period-doubling repeller:

$$h_{n-1}(x)=(\mathscr{L}\circ h_n)(x)=f_0'(f_0^{-1}(x))h_n(f_0^{-1}(x))+f_1'h_n(f_1^{-1}(x)). \tag{4}$$

The nth iterate perturbation for the entire strange set grows exponentially as $(\mathscr{L}^n\circ h)(x)\propto\delta^n$, where δ is the leading eigenvalue of the transfer operator $\mathscr{L}$. For the period-doubling renormalization, δ is the Feigenbaum δ.

The spectrum of $\mathscr{L}$ can be extracted [6, 3] from $\det(1-z\mathscr{L})$ expressed in terms of the traces

$$\operatorname{tr}\mathscr{L}^n=\int \mathrm{d}x\,\delta(x-f^{-n}(x))\prod_{k=0}^{n-1}f'(f^k(x))=\sum_i^{(n)}\frac{\Lambda_i}{1-1/\Lambda_i}$$

$$\begin{aligned}\det(1-z\mathscr{L})&=\exp\left(-\sum_p\sum_{r=1}^{\infty}\frac{z^{rn_p}}{r}\frac{\Lambda_p^r}{1-1/\Lambda_p^r}\right)\\&=1+\sum_{n=1}^{\infty}C_nz^n\end{aligned} \tag{5}$$

where the sum goes over all periodic points x_i of period n, and p are prime (non-repeating) cycles. This Fredholm determinant can also be expressed as a Selberg-type product [12] over all prime cycles:

$$Z(z)=\det(1-z\mathscr{L})=\prod_p\prod_{k=0}^{\infty}(1-z^{n_p}\Lambda_p^{1-k}). \tag{6}$$

The leading zero $z=1/\delta$ of $\det(1-z\mathscr{L})$ corresponds to the Feigenbaum δ. Here Λ_p is the stability of the p cycle; as the map (1) is everywhere expanding, $|\Lambda_p|>1$ for all cycles. The symbolic dynamics is unrestricted binary dynamics; for each binary string i of length n there exists a cycle of stability

$$\Lambda_i=\prod_{k=0}^{n-1}f'(f^k(x_i)).$$

In particular, the stabilities of the fixed points $x_0=1$, $x_1=0$ of (1) are $\Lambda_0=\alpha^2$, $\Lambda_1=\alpha$, where for quadratic maps

$$\alpha=-2.502\,907\,875\,095\,892\,822\,283\,902\,873\ldots.$$

The stabilities of prime cycles up to length 6 are given in table 2 of [4]. Our cycles are computed using Lanford's expansion [13] of $g(x)$; this is the main limitation on the convergence of the present calculation, and, if needed, the accuracy could be improved by recomputing $g(x)$.

The convergence of the $Z(z)=\Sigma_n C_nz^n$ expansion can be estimated as follows. The Fredholm determinant (6) is a product of dynamical zeta functions [6] $1/\zeta_k(z)=\Pi_p(1-z^{m_p}\Lambda_p^{1-k})$. In the piecewise-linear approximation to the repeller (1), we keep only the fixed points and drop all curvature terms in the cycle expansion [3] of $1/\zeta_k=1-(\alpha^{1-k}+(\alpha^2)^{1-k})z-\ldots$. For large k, $1/\zeta_k(z)\approx 1-z\alpha^{1-k}$. In this approximation, the spectrum of the Selberg product (6) is given by

$$Z(z)\approx\prod_{k=0}^{\infty}(1-z\alpha^{1-k})=\sum_{n=0}^{\infty}\alpha^{-n(n-3)/2}\left(\prod_{k=1}^{n}(1-\alpha^{-k})\right)^{-1}(-z)^n. \tag{7}$$

By this simple estimate the eigenvalues δ_k should fall off exponentially as α^{1-k} and the coefficients in the $Z(z)=\Sigma_n C_nz^n$ expansion of the Fredholm determinant (5) should fall off faster than exponentially, as $|C_n| \approx \alpha^{-n(n-3)/2}$. In contrast, the cycle expansions [4] of truncated Selberg products fall off 'only' exponentially; the difference is illustrated in figure 1. The above spectrum of $\mathscr{L}$ for the piecewise-linear approximate map is only indicative of the spectrum for the exact nonlinear map; the details are subtle and the reader is referred to [14] for more careful convergence estimates.

Actual calculation is straightforward. We substitute the eigenvalues of prime cycles up to length N into the Fredholm determinant cycle expansion (5) $Z(z) = \exp(-\Sigma_{n=1}^{N} b_n z^n)$ and then expand the exponential to obtain a polynomial approximation $Z(z) = 1 + \Sigma_{n=1}^{N} C_n z^n$. The zeros can be easily determined by standard numerical methods.

The first twelve eigenvalues are listed in table 1. They agree with the estimates of Eckmann and Epstein†. The odd eigenvalues correspond to the smooth conjugacies and are given by $\delta_k = \alpha^{1-k}$. This fact provides a useful check of the calculation. Conversely, by dividing (5) by the product of the known eigenvalues

$$\prod_{k=0}^{\infty} (1 - z\alpha^{-2k}) = \exp\left(\sum_{n=1}^{\infty} \frac{z^n}{n} \frac{1}{1-\alpha^{-2n}} \right) \tag{8}$$

we obtain the Fredholm determinant for the non-trivial eigenvalues only, with convergence improved by replacement $\alpha \to \alpha^2$ in the estimate (7). This yields more eigenvalues to considerably higher accuracy, see table 1. The convergence of the leading eigenvalue

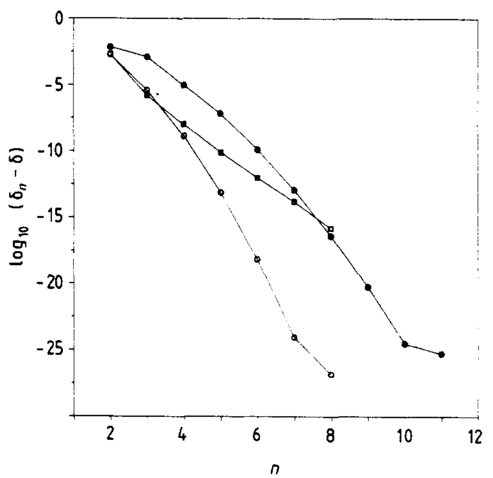

Figure 1. The convergence of the Fredholm determinant cycle expansions (5) for the Feigenbaum δ. Plotted is the estimate $\log_{10}|\delta_{(n)} - \delta|$ error as a function of the cycle length n, where δ is our best estimate of the Feigenbaum δ. Filled dots: $\delta_{(n)}$ computed from the full Fredholm determinant (5). Open dots: $\delta_{(n)}$ computed from the Fredholm determinant with 'trivial' roots (8) divided out. For comparison, the squares are the best estimates of $\delta_{(n)}$ taken from figure 7 of [4] based on truncated $1/\zeta_k$ products, which converge exponentially, in contrast to the α^{-n^2} convergence of the Fredholm determinant expansion. To go beyond cycle length 10 (respectively 8) would require extending the precision of the input function $g(x)$.

† We are grateful to H Epstein and J-P Eckmann for providing us with their unpublished results for the spectrum. The number of eigenvalues obtained here is also comparable to the (unpublished) results of M J Feigenbaum (private communication).

Table 1. Second column: the spectrum of the period doubling operator, computed by including cycles up to length 10 into the Fredholm determinant (5) and determining its zeros. The eigenvalues are numerically stable (and real) to the digits quoted. The leading eigenvalue is the Feigenbaum δ. Third column: deviation of δ_k from α^{1-k}. Since the odd eigenvalues are given exactly by $\delta_{2k+1}=\alpha^{-2k}$, these deviations provide a check on the convergence of the finite cycle length expansions. The even eigenvalues converge toward the asymptotic estimate (7). Fourth column: the even eigenvalues computed from the Fredholm determinant with 'trivial' roots (8) divided out. Last column: the eigenvalues obtained through diagonalization of the transfer operator in the polynomial representation.

k	δ_k	$\delta_k-\alpha^{1-k}$	δ_k, k even	δ_k
0	4.669 201 609 102 99			4.669 201 609 102 99
1	1.000 000 000 000 00	2.5×10^{-21}		1.0
2	−0.123 652 712 553		−0.123 652 712 552 6870	$-1.236\,527\,125\,526\,87\times10^{-1}$
3	0.159 628 440 3827	-3.1×10^{-15}		$1.596\,284\,403\,826\,99\times10^{-1}$
4	−0.057 307 0211	6.5×10^{-3}	−0.057 307 021 066 68	$-5.730\,702\,106\,668\,18\times10^{-2}$
5	0.025 481 239	1.5×10^{-10}		$2.548\,123\,897\,901\,31\times10^{-2}$
6	−0.010 145 8	-3.5×10^{-5}	−0.010 145 805 67	$-1.014\,580\,567\,208\,85\times10^{-6}$
7	0.004 067	-2.9×10^{-7}		$4.067\,530\,437\,238\,72\times10^{-3}$
8	−0.001 63	-1.6×10^{-7}	−0.001 625 278	$-1.625\,278\,165\,366\,60\times10^{-3}$
9	0.0007	2.3×10^{-5}		$6.492\,935\,399\,055\,78\times10^{-4}$
10		-6.3×10^{-7}	−0.000 2588	$-2.587\,772\,471\,663\,93\times10^{-4}$
12		-4.1×10^{-7}	−0.000 041	$-4.131\,118\,010\,446\,64\times10^{-5}$

as a function of the maximal cycle length is shown in figure 1—only 71 cycles up to length 8 suffice to determine δ to 25 significant figures.

Given the cycle eigenvalues, one can with equal ease evaluate other averages associated with the repeller. For example, the escape rate [15] γ_1 and the correlation exponents $\gamma_n-\gamma_1$ are given by [1, 3] the eigenvalues of the operator $\mathscr{L}(y,x)=\delta(y-f(x))$, i.e. the zeros $z=\mathrm{e}^{\gamma}$ of the determinant [3]:

$$\det(1-z\mathscr{L})=\prod_{k=0}^{\infty}\prod_{p}\left(1-\frac{z^{n_p}}{|\Lambda_p|\Lambda_p^k}\right). \tag{9}$$

The Hausdorff dimension D_{H} can likewise be extracted [3] from the leading zero $\tau=-D_{\mathrm{H}}$ of the Fredholm determinant for the transfer operator $\mathscr{L}_\tau(x,y)=\delta(x-f^{-1}(y))|f'(x)|^\tau$:

$$\det(1-\mathscr{L}_\tau)=\prod_{k=0}^{\infty}\prod_{p}\left(1-\frac{|\Lambda_p|^\tau}{\Lambda_p^k}\right). \tag{10}$$

The convergence of the eigenvalue spectrum of (9) and (10) is comparable to that of the stability spectrum (6), and all the eigenvalues computed are again real.

As an independent check, we also compute the spectrum by exploiting the analyticity of the map to represent the linearized period-doubling operator in terms of even polynomial basis vectors (i.e. with the trivial α^{1-2k} eigenvalues excluded)

$$e_n(x)=(x^2-a)^n \qquad \mathscr{L}e_n=e_mL_{mn}.$$

Setting $a=0.7$ (the spectrum is rather insensitive to the precise value of a) and diagonalizing the [50×50] truncation of L_{mn}, we obtain the first 50 eigenvalues to the machine precision. As the eigenvalues fall off as α^{1-k}, the zeroth eigenvalue is given to some 30 digits, with the number of significant digits falling off to one digit by the

49th eigenvalue. All the computed eigenvalues are real within the accuracy of the calculation (though we do not know why this should be so).

To summarize: cycle expansions of the Fredholm determinants applied to the period-doubling repeller converge extremely well. Including cycles up to length 8, we obtain our best estimate of the Feigenbaum δ, the Hausdorff dimension and the escape rate for the period-doubling repeller,

$$\begin{aligned} \delta &= 4.669\,201\,609\,102\,990\,671\,853\,2038\ldots \\ D_{\mathrm{H}} &= 0.538\,045\,143\,580\,549\,911\,671\,415\,567\ldots \\ \gamma &= 0.554\,613\,533\,486\,294\,443\,341\,193\,309\ldots \end{aligned} \tag{11}$$

(all numbers stated here are numerically stable to the digits quoted). The Hausdorff dimension estimate has nearly three times as many significant figures as the most accurate estimates available in the literature [6, 7, 18, 19], in agreement with the recent claims [20]. The direct diagonalization of the transfer operator yields an even better spectrum, but an implementation in a more general setting might be less straightforward than the cycle expansion approach. The point of the above exercises is not so much the pleasure of owning D_{H} to 30 digits, as developing the confidence in the cycle expansions to be used in contexts where convergence is much harder to check and where the entire spectrum is of physical interest, such as in the evaluation of quantum spectra of classically chaotic systems.

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References

[1] Christiansen F, Paladin G and Rugh H H 1990 *Phys. Rev. Lett.* submitted
[2] Sullivan D 1989 *Universality in Chaos* 2nd edn, ed P Cvitanović (Bristol: Hilger)
[3] Artuso R, Aurell E and Cvitanović P 1990 *Nonlinearity* **3** 325
[4] Artuso R, Aurell E and Cvitanović P 1990 *Nonlinearity* **3** 361
[5] Jiang Y and Sullivan D 1989 *preprint* CUNY
[6] Ruelle D 1978 *Statistical Mechanics, Thermodynamic Formalism* (Reading, MA: Addison-Wesley)
[7] Aurell E 1990 *J. Stat. Phys.* to appear
[8] Aurell E unpublished
[9] Feigenbaum M J 1988 *J. Stat. Phys.* **52** 527
[10] Feigenbaum M J 1979 *J. Stat. Phys.* **21** 669
[11] Feigenbaum M J unpublished
[12] Selberg A 1956 *J. Ind. Math. Soc.* **20** 47
[13] Lanford III O E 1982 *Bull. Am. Math. Soc.* **6** 427
[14] Ruelle D 1976 *Invent. Math.* **34** 231
[15] Kadanoff L and Tang C 1984 *Proc. Natl Acad. Sci., USA* **81** 1276
[16] Grassberger P 1981 *J. Stat. Phys.* **26** 173
[17] Bensimon D, Jensen M H and Kadanoff L P 1986 *Phys. Rev.* A **33** 3622
[18] Aurell E 1986 *Phys. Rev.* A **34** 5135; 1987 **35** 4016
[19] Kovács Z 1989 *J. Phys. A: Math. Gen.* **22** 5161
[20] Grassberger P, Kantz H and Moening U 1989 *J. Phys. A: Math. Gen.* A **22** 5217